Definition : Set

A set is an unordered collection of distinct objects which are called elements. A set must be well defined.

Definition : Well Defined

A set is well defined if when given an object, you can determine if the object is in the set or not.

Definition : Injective / One-to-one / 1-1

A function $f : A \to B$ is injective if $f(a) = f(b)$ forces $a = b$.

Remark : When is a function not injective?

A function is not injective when there exists a, b with $f(a) = f(b)$, and $a \neq b$.

Definition : Surjective / Onto

A function $f : A \to B$ is surjective if $f(A) = B$ (Note that $f(A)$ is the image, $f(A) = \{f(a) : a \in A\}$). That is, for all $b \in B$, there exists $a \in A$ with $f(a) = b$.

Remark : When is a function not surjective?

A function is not surjective when there exists $b \in B$ such that for all $a \in A$, $f(a) \neq b$.

Definition : Image of a subset

Suppose $f : X \to Y$, and let $A \subseteq X$. The image of the subset A is the set obtained by applying the function f to every element of A. This is denoted by $f(A)$.

$$f(A) = \{f(a) : a \in A\}$$

Definition : Preimage of a Subset

Suppose $f : X \to Y$, and let $B \subseteq Y$. The preimage of B is everything in X that gets mapped to B by f. This is denoted by $f^{-1}(B)$.

$$f^{-1}(B) = \{x \in X : f(x) \in B\} \subseteq X$$

Definition : Bijection

A function that is both injective and surjective is a bijection. This means that every element in A gets mapped to a unique element in B, and everything in B has a solution in A. If we have $f : X \to Y$, and f is a bijection, then we can define an inverse function $f^{-1} : B \to A$ where $f^{-1}(b) = a \Leftrightarrow f(a) = b$.

Definition : Field

A field is a nonempty set $\mathbb{F}$ along with 2 binary operations, addition $(+)$ and multiplication $(\cdot)$, satisfying the following axioms:

1. If $a, b \in \mathbb{F}$, then
$$\left.\begin{aligned} a+b &= b+a \\ a\cdot b &= b\cdot a \end{aligned}\right\} \text{ Operations are commutative}$$
2. If $a, b, c \in \mathbb{F}$, then
$$a\cdot(b+c) = a\cdot b + a\cdot c \qquad \text{Distributive property}$$
3. If $a, b, c \in \mathbb{F}$, then
$$\left.\begin{aligned} a+(b+c) &= (a+b)+c \\ a\cdot(b\cdot c) &= (a\cdot b)\cdot c \end{aligned}\right\} \text{ Operations are associative}$$
4. There are elements $0, 1 \in \mathbb{F}$ such that for all $a \in \mathbb{F}$:
$$a+0=a \qquad 0 \text{ is the additive identity}$$
$$a\cdot 1=a \qquad 1 \text{ is the multiplicative identity}$$
5. For each $a \in \mathbb{F}$ there is $-a \in \mathbb{F}$ such that:
$$a+(-a)=0 \qquad \text{Additive inverse}$$
For each $a \in \mathbb{F}$ with $a \neq 0$, there is $a^{-1} \in \mathbb{F}$ such that:
$$a\cdot a^{-1}=1 \qquad \text{Multiplicative inverse}$$

Definition : Ordered Field

An ordered field is a field $\mathbb{F}$ along with:

6. There is a nonempty subset $P \subset \mathbb{F}$ called the positive elements such that:
   (a) If $a, b \in P$ then $a+b \in P$ and $a\cdot b \in P$
   (b) If $a \in \mathbb{F}$ and $a \neq 0$, then $a \in P$ or $-a \in P$ but not both. (This is called order)
   0 is its own inverse, so $0 = -0$. For all $a \neq 0$, $a \neq -a$ so $0 \notin P$.

Definition : Order Relation

Let $\mathbb{F}$ be an ordered field with $a, b \in \mathbb{F}$. We say "$a < b$" (a is less than b) if $b - a \in P$, and "$a \leq b$" (a is less than or equal to b) if either $a = b$ or $a < b$.

Definition : Absolute Value

If $\mathbb{F}$ is an ordered field, define the absolute value function $|\quad| : \mathbb{F} \to \mathbb{F}$.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Proposition

If $\mathbb{F}$ is an ordered field and $a, b \in \mathbb{F}$, then $|a| \leq b$ if and only if $-b \leq a \leq b$.

Theorem : The Triangle Inequality

If $x, y \in \mathbb{F}$, then

$$|x + y| \leq |x| + |y|$$

Corollary : Reverse Triangle Inequality

$$||x| - |y|| \leq |x - y|$$

Corollary

$$|x - y| \leq |x| + |y|$$

Corollary

$$|x + y| \geq ||x| - |y||$$

Definition : Distance Function / Metric

A function $d : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ is called a distance function, or metric, if it satisfies:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(x, y) = d(y, x)$ (symmetry)
4. $d(x, z) \leq d(x, y) + d(y, z)$

Definition : Metric Spaces

A metric space is a space that has a metric. This allows us to talk about limits and continuity in terms of the metric.

Definition : Bounds

Let S be an ordered field, and let $A \subseteq S$ be nonempty.

(i) A is bounded above if $\exists b \in S$ such that $\forall x \in A$, $x \leq b$. b is called an upper bound of A.

(ii) The least upper bound of A (if it exists) is some $b_0 \in S$ such that b_0 is an upper bound of A, and if b is any other upper bound of A, then $b_0 \leq b$. b_0 is called the supremum of A, denoted by $\sup(A)$.

(iii) A is bounded below if $\exists c \in S$ such that $\forall x \in A$, $c \leq x$. c is called a lower bound of A.

(iv) The greatest lower bound of A (if it exists) is some $c_0 \in S$ such that c_0 is a lower bound of A, and if c is any other lower bound of A, then $c_0 \geq c$. c_0 is called the infimum of A, denoted by $\inf(A)$.

(v) If a set is both bounded above and bounded below, the set is called bounded.

Definition : Least Upper Bound Property

Let S be an ordered field, then S has the least upper bound property if given any nonempty set, $A \subseteq S$ where A is bounded above, A has a least upper bound in S. That is, $\sup(A) \in S$ exists.

A set that has the least upper bound property is complete.

Theorem : Existence and Uniqueness of $\mathbb{R}$

There exists a unique complete ordered field, we call this field the real numbers.

We could relabel the elements, but the field would be isomorphic. The construction uses Dedekind cuts.

Proposition : Uniqueness of the sup and inf

If $\sup(A)$ or $\inf(A)$ of some $A \subseteq \mathbb{R}$ exists, it is unique.

Theorem : Supremum and Infimum Characterization

Let $A \subseteq \mathbb{R}$, $\sup(A) = \alpha$ if and only if:

(i) α is an upper bound of A

(ii) Given any $\varepsilon > 0$, $\sup(A) - \varepsilon = \alpha - \varepsilon$ is not an upper bound of A. That is, there is some $x \in A$ such that $\alpha - \varepsilon < x$.

Likewise, $\inf(A) = \beta$ if and only if:

(i) β is a lower bound of A

(ii) Given any $\varepsilon > 0$, $\inf(A) + \varepsilon = \beta + \varepsilon$ is not a lower bound of A. That is, there is some $x \in A$ such that $x < \beta + \varepsilon$.

Lemma : Archimedean Principle

If $a, b \in \mathbb{R}$ with $a > 0$ then there exists a rational number n such that

$$na > b \Leftrightarrow n > \frac{b}{a}$$

Definition : Dense Sets

Suppose A and B are subsets of ordered fields. Then A is dense in B if for every $x, y \in B$ with $x < y$, there exists $a \in A$ such that $x < a < y$.

Definition : Maximal and Minimal Elements

Let $A \subseteq \mathbb{R}$, A has a maximal element if there exists $M \in A$ such that $x \leq M$ for all $x \in A$.
Likewise, A has a minimal element if there exists $m \in A$ such that $m \leq x$ for all $x \in A$.

Lemma

Let $x, y \in \mathbb{R}$, where $y - x > 1$. Then, there exists a $z \in \mathbb{Z}$ such that $x < z < y$.

Theorem

$\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition : Ceiling

The ceiling of x, denoted $\lceil x \rceil$ is the integer n such that

$$x \leq n < x + 1$$

Definition : Floor

The floor of x, denoted $\lfloor x \rfloor$ is the integer n such that

$$x - 1 < n \leq x$$

Definition : Closed Interval

A closed interval between a and b is the interval containing the endpoints a and b. Denoted by:

$$[a, b] = \{x : a \leq x \leq b\}$$

Definition : Open Interval

An open interval between a and b is the interval that doesn't contain the endpoints a and b. Denoted by:

$$(a, b) = \{x : a < x < b\}$$

Definition : Well Ordering Principle

Every nonempty subset of $\mathbb{N}$ has a smallest element.

Definition : Same Size

Two sets have the same size if and only if there is a bijection between them.

Definition : Equivalent Sets

We define a relation on sets by $A \sim B$ if there exists a bijection from A to B. This is an equivalence relation.

Definition : Equivalence Relation

An equivalence relation is a binary relation that satisfies:

1. Reflexivity: $A \sim A$
   - Using identity, $i_A : A \to A$, $i_A(x) = x$
2. Symmetry: If $A \sim B$, then $B \sim A$
   - Since bijections are invertible
3. Transitivity: If $A \sim B$, and $B \sim C$, then $A \sim C$
   - By composition of bijections

Definition : Infinite Set

One way to define what it means to be an infinite set is that the set is the same as a proper subset of itself.

Definition : Countable Sets

A countable set is either finite, or is equivalent to $\mathbb{N}$. If $|A| = |\mathbb{N}|$, then we say A is countably infinite, or listable. We can write $A = \{a_1, a_2, a_3, \dots\}$

Theorem

$|\mathbb{R}| > |\mathbb{N}|$

Definition : Continuum Hypothesis

There is no set whose cardinality is strictly between $|\mathbb{N}|$ and $|\mathbb{R}|$.

Based on the axioms of set theory, this is unprovable.

Theorem

If A is a set and $\mathcal{P}(A)$ is the power set of A, then $|A| < |\mathcal{P}(A)|$

Definition : Sequences

A sequence of real numbers is a function $a : \mathbb{N} \to \mathbb{R}$. We usually write a_n instead of $a(n)$. The notation is:

$$\{a_n\}_{n=1}^{\infty} \qquad \text{or} \qquad \{a_n\} \qquad \text{or} \qquad (a_1, a_2, \dots)$$

Sometimes, we have a formula, ex. $a_n = \frac{1}{n}$ or $a_n = n^2$. Sometimes the function is recursive, ex. $a_n = a_{n-1} + a_{n-2}$.

Definition : Bounded Sequences

$\{a_n\}$ is bounded if the range $\{a_n : a \in \mathbb{N}\}$ is bounded. This means there exists $L, U \in \mathbb{R}$ such that

$$L \leq a_n \leq U \qquad \text{for all } n \in \mathbb{N}$$

Proposition

$\{a_n\}$ is bounded if and only if there exists a $C \in \mathbb{R}$ with $|a_n| \leq C$ for all $n \in \mathbb{N}$.

Definition : Convergent Sequence

(a_n) converges to $a \in \mathbb{R}$ if $\forall\, \varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ when $n > N$. a is called the limit of a_n.

$$\begin{aligned} |a_n - a| < \varepsilon &\Leftrightarrow -\varepsilon < a_n - a < \varepsilon \\ &\Leftrightarrow a - \varepsilon < a_n < a + \varepsilon \\ &\Leftrightarrow a \in (a - \varepsilon, a + \varepsilon) \end{aligned}$$

That is, for some $N \in \mathbb{R}$ and all $n > N$, we are far enough in the sequence to stay close within the open interval $(a - \varepsilon, a + \varepsilon)$,

Definition : Divergent Sequence

If a sequence doesn't converge, then it diverges. There are 3 forms of this:

1. (a_n) diverges to $+\infty$

   $\lim_{n\to\infty} a_n = +\infty$ if for all $M \in \mathbb{R}$ there is some $N \in \mathbb{R}$ such that $a_n > M$ for $n > N$. (Eventually the sequence stays bigger than M).

2. (a_n) diverges to $-\infty$

   $\lim_{n\to\infty} a_n = -\infty$ if for all $M \in \mathbb{R}$ there is some $N \in \mathbb{R}$ such that $a_n < M$ for $n > N$. (Eventually sequence stays smaller than M).

3. Limit doesn't exist

   We need to negate the definition: a_n does not converge to $a \in \mathbb{R}$ if there exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$.

   $$|a_n - a| \geq \varepsilon \qquad \text{for some } n > N$$

Proposition

If a sequence converges, the limit is unique.

Proposition

If a sequence is convergent, it is bounded.

Laws : Limit Laws

Assume $a_n \to a$, $b_n \to b$, $a, b, c \in \mathbb{R}$

1. $(a_n + b_n) \to a + b$
2. $(a_n - b_n) \to a - b$
3. $(a_n \cdot b_n) \to a \cdot b$
4. $\left(\frac{a_n}{b_n}\right) \to \frac{a}{b}$ $\qquad$ provided $b_n \neq 0$ for all n, $b \neq 0$
5. $(c \cdot a_n) \to c \cdot a$

Theorem : Squeeze Theorem

Assume $a_n \leq x_n \leq b_n$ for all n and $a_n \to L$ and $b_n \to L$, then $x_n \to L$.

Definition : Monotone Increasing and Monotone Decreasing

A sequence (a_n) is monotone increasing if

$$a_n \leq a_{n+1} \qquad \text{for all } n$$

Likewise, a sequence (a_n) is monotone decreasing if

$$a_n \geq a_{n+1} \qquad \text{for all } n$$

Definition : Monotone

If a sequence (a_n) is either monotone increasing or monotone decreasing, we say that it is monotone.

Theorem : Monotone Convergence Theorem (MCT)

Suppose (a_n) is monotone. Then (a_n) converges if and only if it is bounded. Moreover, if (a_n) is increasing, then (a_n) either diverges to $+\infty$ or $\lim_{n\to\infty} a_n = \sup(\{a_n : n \in \mathbb{N}\})$. If (a_n) is decreasing, then either it diverges to $-\infty$ or $\lim_{n\to\infty} a_n = \inf(\{a_n : n \in \mathbb{N}\})$.

Proposition

Suppose $S \subseteq \mathbb{R}$ is bounded above. Then, there exists a sequence (a_n) where $a_n \in S$ for all n and $\lim_{n\to\infty} a_n = \sup(S)$

Likewise if S is bounded below, there is a sequence (b_n) where each $b_n \in S$ and $b_n \to \inf(S)$

Definition : Subsequences

Let (a_n) be a sequence. Let $n_1 < n_2 < n_3 < \ldots$ be an increasing sequence of natural numbers. Then, $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ is called a subsequence of (a_n) and is denoted a_{n_k}.

Proposition

A sequence converges to a if and only if every subsequence converges to a. (This is useful for showing a sequence diverges).

Corollary

If (a_n) has a pair of subsequences that converge to different limits, then (a_n) diverges.

Proposition

If (a_n) is a monotone sequence that has a convergent subsequence, then (a_n) converges to the same limit.

Lemma

Every sequence has a monotone subsequence.

Theorem : Bolzano-Weierstrass

Every bounded sequence has a convergent subsequence. *The idea is bounded and monotone gives convergence.*

Definition : Cauchy Sequences

A sequence (a_n) is Cauchy if for every $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that $|a_n - a_m| < \varepsilon$ for $n, m > N$.

Definition : Series

The sum of infinitely many numbers is called a series.

Proposition : k^{th} Term Test

If $a_k \not\to 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Contrapositive: If $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \to 0$.

Proposition : Geometric Series Test

$$\sum_{k=0}^{\infty} ar^k = \begin{cases} \frac{a}{1-r} & \text{if } r \in (-1, 1) \\ \text{diverges} & \text{if } |r| \geq 1 \end{cases}$$

Proposition

If $a_k \geq 0$ for all k, then $\sum a_k$ either converges, or diverges to $+\infty$.

Proposition : Comparison Test

Assume $0 \leq a_k \leq b_k$ for all k.

1. If $\sum b_k$ converges, then so does $\sum a_k$
2. If $\sum a_k$ diverges, then so does $\sum b_k$

Proposition

The Harmonic Series diverges.

Proposition : p-Series

$\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if and only if $p > 1$.

Proposition : Alternating Series Test

Assume (a_k) is monotonically decreasing with $a_k \to 0$, then

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$$

converges.

Note: a_k is decreasing with a limit of 0, which implies each $a_k \geq 0$.

Proposition

If $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ also converges.

Definition : Absolute Convergence

If $\sum_{k=1}^{\infty} |a_k|$ converges, we say $\sum_{k=1}^{\infty} a_k$ converges absolutely.

If $\sum_{k=1}^{\infty} a_k$ converges, but $\sum_{k=1}^{\infty} |a_k|$ does not, we say $\sum_{k=1}^{\infty} a_k$ converges conditionally.

Definition : Continuity

A function $f : A \to \mathbb{R}$ is continuous at a point $c \in A$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in A$ with $|x - c| < \delta$ we have $|f(x) - f(c)| < \varepsilon$.

This means, for $x \in A$ with $x \in (c - \delta, c + \delta)$, we have $f(x) \in (f(c) - \varepsilon, f(c) + \varepsilon)$. Note that we include $x = c$

If $f(x)$ is continuous at every point in its domain, we say it is continuous.

Definition : Negation of Continuity at $x = c$

A function $f(x)$ is not continuous at $x = c$ if there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exists x with $|x - c| < \delta$, but $|f(x) - f(c)| \geq \varepsilon$

Definition : Dirichlet Function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Definition : Modified Dirichlet Function

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Definition : Limit Point

A point x is a limit point of a set A if there is a sequence of points $a_1, a_2, a_3, \ldots$ from $A \setminus \{x\}$ such that $a_n \to x$

A point x is a limit of point A iff every ε-neighbourhood of x intersects A at some point other than x

Definition : Epsilon Neighbourhood

The ε-neighbourhood of x is $(x - \varepsilon, x + \varepsilon)$ for some $\varepsilon > 0$.

Definition : Functional Limit

Let $f : A \to \mathbb{R}$ and c is a limit point of A, then $\lim_{x \to c} f(x) = L$ if for all $\varepsilon > 0$, $\exists\, \delta > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$.

Definition : Negation of Functional Limit

There exists $\varepsilon > 0$, such that for all $\delta > 0$ there is some $x \in A$ with $0 < |x - c| < \delta$ and $|f(x) - L| \geq \varepsilon$.

Proposition

$\lim_{x \to c} f(x)$ can converge to at most 1 value. That is, if the limit exists, then it is unique.

Theorem

Let $A \subseteq \mathbb{R}$, c is a limit point of A, then $\lim_{x \to c} f(x) = L$ if and only if for every sequence (a_n) for which each $a_n \in A$ with $a_n \neq c$ and $a_n \to c$, we have $f(a_n) \to L$.

Laws : Functional Limit Laws

Let f, g be functions from $A \subseteq \mathbb{R}$ to $\mathbb{R}$, c is a limit point of A, and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M$$

1. $\lim_{x \to c} k \cdot f(x) = k \cdot L$ for any $k \in \mathbb{R}$
2. $\lim_{x \to c} [f(x) + g(x)] = L + M$
3. $\lim_{x \to c} [f(x) \cdot g(x)] = L \cdot M$
4. $\lim_{x \to c} \left[\frac{f(x)}{g(x)}\right]$ provided $M \neq 0$ and $g(x) \neq 0$ for $x \in A$
5. *To prove this, use the sequence limit laws*

Theorem : Squeeze Theorem for Functions

For $f, g, h : A \to \mathbb{R}$, c is a limit point of A and $f(x) \leq g(x) \leq h(x)$ for all $x \in A$, and $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$, then $\lim_{x \to c} g(x) = L$

Proposition

Let $A, B \subseteq \mathbb{R}$, $g : A \to B$ is continuous at c, $f : B \to \mathbb{R}$ is continuous at $g(c)$, then $(f \circ g)(x) : A \to \mathbb{R}$ is continuous at c

Definition : Delta Neighbourhood

Given $\delta > 0$, the δ-neighbourhood of a point $x \in \mathbb{R}$ is denoted by $V_\delta(x) = (x - \delta, x + \delta) = \{y \in \mathbb{R} : |x - y| < \delta\}$

Definition : Open Set

A set $U \subseteq \mathbb{R}$ is open if for all $x \in U$, $\exists\, \delta > 0$ such that $(x - \delta, x + \delta) \subseteq U$. That is, if $\forall\, x \in U$, $\exists\, V_\delta(x) \subseteq U$

This is equivalent to saying $U \subseteq \mathbb{R}$ is open if $\forall\, x \in U$, $\exists\, \delta > 0$ such that if $|x - y| < \delta$, then $y \in U$

Proposition : (5.3)

(i) If $\{U_\alpha\}$ is any collection of open sets, then $\bigcup_\alpha U_\alpha$ is also an open set

(ii) If $\{U_1, U_2, \ldots, U_n\}$ is a finite collection of open sets, then $\bigcap_{k=1}^{n} U_k$ is also an open set

We can do arbitrary unions and finite intersections of open sets, and end up with an open set

Theorem : (5.5)

Every open set is a countable union of disjoint open intervals

Definition : (5.6)

A set $A \subseteq \mathbb{R}$ is closed if A^c is open.

This is equivalent to saying A is closed if there exists an open set B such that $B^c = A$

Theorem : (5.10)

A set A is closed iff it contains all of its limit points

Fact : (5.11)

$$\left(\bigcup_{\alpha} U_\alpha\right)^c = \bigcap_{\alpha} U_\alpha^c \quad \text{and} \quad \left(\bigcap_{\alpha} U_\alpha\right)^c = \bigcup_{\alpha} U_\alpha^c$$

Definition : Topological Space

A topological space is a pair (X, τ) of a set X and a collection τ of subsets of X (these are the opens sets) satisfying:

1. Arbitrary union of open sets is open
2. Finite intersection of open sets is open
3. X and $\emptyset$ are open

Example : (Exercise 5.7)

x is a limit point of A iff every ε-neighbourhood of x intersects A at some point other than x

Proposition

1. If $\{U_1, U_2, \ldots, U_n\}$ are closed sets, then $\bigcup_{k=1}^{\infty} U_k$ is also closed
2. If $\{U_\alpha\}_{\alpha \in I}$ is a collection of closed sets, then $\bigcap_{\alpha \in I} U_\alpha$ is closed

That is, we can do finite unions and arbitrary intersections of closed sets to get a closed set. This follows from the fact that we can do arbitrary unions and finite intersections of open sets to get an open set.

Definition : Covers

Let A be a set. The set $\{U_\alpha\}_{\alpha \in S}$ is a cover of A if $A \subseteq \bigcup_{\alpha \in S} U_\alpha$

If each U_α is open, we say it is an open cover

If $\{U_\alpha\}_{\alpha \in S}$ has a finite subset $\{U_\alpha\}_{\alpha \in F}$ (assuming $F \subseteq S$) which is still a cover of A, then $\{U_\alpha\}_{\alpha \in F}$ is a finite subcover

Definition : Compact Set

A set A is compact if every open cover of A contains a finite subcover.

For example, $(0, 4)$ is not compact, but $[0, 4]$ is.

Theorem : Heine-Borel

A set $S \subseteq \mathbb{R}$ is compact iff it is closed and bounded.

Theorem : Equivalent Forms of Continuity

The following are equivalent:

(i) f is continuous at c

(ii) $\forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $\forall\, x \in A$ with $|x - c| < \delta$ we have $|f(x) - f(c)| < \varepsilon$

(iii) For any ε-neighbourhood of $f(c)$ denoted by $V_\varepsilon(f(c))$, there exists some δ-neighbourhood of c denoted by $V_\delta(c)$ with the property that $\forall\, x \in A$ with $x \in V_\delta(c)$, we have $f(x) \in V_\varepsilon(f(c))$

(iv) For all sequences (a_n) from A which converge to c ($\lim_{n\to\infty} a_n = c$) we have $(f(a_n))$ converging to $f(c)$. That is, $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$

(v) $\lim_{x\to c} f(x) = f(c)$

Theorem : Topological Continuity

Let $f : X \to \mathbb{R}$, then f is continuous if and only if for every open set $B \subseteq \mathbb{R}$, $f^{-1}(B) = A \cap X$ for some open set A.

Theorem

Suppose $f : X \to \mathbb{R}$ is continuous. If $A \subseteq X$ is compact, then $f(A)$ is compact.

Corollary

The image of a compact set under a continuous function is bounded (and also closed) by Heine Borel.

Theorem : Extreme Value Theorem (EVT)

A continuous function on a compact set attains its sup and inf. That is, there is some a_0, b_0 in the set such that $f(a_0) =$ sup of the image, $f(b_0) =$ inf of the image.

Definition : Interior

The interior of a set A, denoted by $\text{Int}(A)$, is the set of points x such that there is an open neighbourhood of x that is a subset of A.

Definition : Exterior

The exterior of a set A, denoted by $\text{Ext}(A)$, is the set of points x such that there is an open neighbourhood of x that is a subset of A^c. That is, $\text{Ext}(A) = \text{Int}(A^c)$.

Definition : Boundary

The boundary of a set A, denoted by ∂A, is the set of points x such that every neighbourhood of x contains points from A and A^c.

Note

We could show that for any set A,

$$\mathbb{R} = \text{Int}\,(A) \cup \partial A \cup \text{Ext}\,(A)$$

where each of these sets are disjoint.

Lemma

If f is continuous, and $f(c) > 0$, then there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in (c - \delta, c + \delta)$.

Likewise, if $f(c) < 0$, we can find $\delta > 0$ such that for all $x \in (c - \delta, c + \delta)$ we have $f(x) < 0$.

Proposition

If f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have different signs (one positive, one negative), then there is some $c \in (a, b)$ for which $f(c) = 0$.

Theorem : Intermediate Value Theorem (IVT)

If f is continuous on $[a, b]$ and α is any number between $f(a)$ and $f(b)$. Then there exists $c \in (a, b)$ for which $f(c) = \alpha$

Definition : Uniform Continuity

$f : A \to \mathbb{R}$ is uniformly continuous if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in A$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

Proposition

If $f : A \to \mathbb{R}$ is continuous and A is compact, then f is uniformly continuous on A

Definition : Differentiability

Let I be an interval, $f : I \to \mathbb{R}$, and $c \in I$. We say f is differentiable at $c \in I$ if the following limit exists

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

If C is a collection of points where f is differentiable, then the derivative of f is a function $f' : C \to \mathbb{R}$, where

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Theorem : S

ppose I is an interval and $f : I \to \mathbb{R}$ is differentiable at $c \in I$. Then, f is continuous at c.

Laws : Differentiability Laws

Let I be an interval, $k \in \mathbb{R}$, and $f, g : I \to \mathbb{R}$ where both f and g are differentiable at $c \in \mathbb{R}$.

1. $(f + g)'(c) = f'(c) + g'(c)$
2. $(f - g)'(c) = f'(c) - g'(c)$
3. $(k \cdot f)'(c) = k \cdot f'(c)$

Theorem : Product Rule

$(fg)'(c) = f'(c) \cdot g(c) + f(c) \cdot g'(c)$

Theorem : Chain Rule

Let I_1, I_2 be intervals, $g : I_1 \to I_2$, $f : I_2 \to \mathbb{R}$, $(f \circ g) : I_1 \to \mathbb{R}$, g is differentiable at $c \in I_1$, f is differentiable at $g(c) \in I_2$, then $f \circ g$ is differentiable at c and $(f \circ g)' = f'(g(c)) \cdot g'(c)$

Definition : Local Minima and Maxima

$f : A \to \mathbb{R}$ has a local max at $c \in A$i f there exists $\delta > 0$ such that for all $x \in A$ with $|x - c| < \delta$, we have $f(x) \leq f(c)$

Likewise, f has a local min at c if there exists $\delta > 0$ such that for all $x \in A$ with $|x - c| < \delta$ we have $f(x) \geq f(c)$

Note

Recall: A sequence (S_n) where each term $S_n \geq 0$ and $S_n \to L$, then $L \geq 0$

Proposition

Let I be an open interval and f is differentiable at $c \in I$. If f has a local max or min at c, then $f'(c) = 0$

Note : Intermediate Value Property (IVP)

A function has the intermediate value property on $[a, b]$ if when α is any value between $f(a)$ and $f(b)$ there is some $c \in (a, b)$ with $f(c) = \alpha$

Theorem : Darboux's Theorem

Let f be differentiable on (a, b). If α is between $f'(a)$ and $f'(b)$ then there exists $c \in (a, b)$ where $f'(c) = \alpha$

Theorem : Rolle's Theorem

$f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there exists $c \in (a, b)$ with $f'(c) = 0$

Theorem : Mean Value Theorem (MVT)

$f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). There exists $c \in (a, b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$. I.e., $f'(c)(b-a) = f(b) - f(a)$

Corollary

Let I be an interval and $f : I \to \mathbb{R}$ is differentiable. If $f'(x) = 0$ for all $x \in I$ then f is constant.

Corollary

Let I be an interval, $f, g : I \to \mathbb{R}$, both differentiable on I. If $f'(x) = g'(x)$ for all $x \in I$ then $f(x) = g(x) + C$ for some $c \in \mathbb{R}$.

Corollary

Let I be an interval, $f : I \to \mathbb{R}$, f is differentiable on I

(i) f is monotonic increasing iff $f'(x) \geq 0$ for all $x \in I$

(ii) f is monotonic decreasing iff $f'(x) \leq 0$ for all $x \in I$

Theorem : Cauchy MVT

If f, g are continuous on $[a, b]$ and differentiable on (a, b) then there exists $c \in (a, b)$ such that

$$[f(b) - f(a)] \cdot g'(c) = [g(b) - g(a)] \cdot f'(c)$$

Note: The MVT is a special case of the Cauchy MVT with $g(x) = x$

Theorem : L'Hôpital's Rule

Suppose I is an interval containing a point a and $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are differentiable (except possibly at a) and suppose $g'(x) \neq 0$ on I. Then, if $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ or if $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided the limit exists.

Definition : Lipschitz Continuity

f is called Lipschitz continuous if there exists $K \in \mathbb{R}$ where $|f(x) - f(y)| \leq K \cdot |x - y|$ for all x, y where f is defined.

Definition : Contraction Mapping

A contraction mapping is a Lipschitz function where $K \in [0, 1)$ so $|f(x) - f(y)| < |x - y|$.

Definition : Iterated Function System (IFS)

An iterated function system (IFS) is a set of contraction mappings.

Definition : Partition

A partition of $[a, b]$ is a finite set $\{x_0, x_1, \ldots, x_n\}$ such that $x_0 = a$, $x_n = b$, $x_0 < x_1 < \cdots < x_n$.

Notion: Given a partition, the i^{th} subinterval is $[x_{i-1}, x_i]$

Definition

$$m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$
$$M_i := \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

Definition : Upper & Lower Sums

Consider $f : [a, b] \to \mathbb{R}$, $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$

$$U(f, P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) \qquad \text{is the upper sum}$$

$$L(f, P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \qquad \text{is the lower sum}$$

Definition : Refinement

Consider a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$. A partition Q of $[a, b]$ is called a refinement of P if $P \subseteq Q$.

Proposition

Let $f : [a, b] \to \mathbb{R}$ and $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. If Q is a refinement of P then,

$$U(f, P) \geq U(f, Q)$$
$$L(f, P) \leq L(f, Q)$$

Proposition

Let $f : [a, b] \to \mathbb{R}$

(a) if P is a partition of $[a, b]$ then $L(f, P) \leq U(f, P)$

(b) if P_1, P_2 are any partitions then $L(f, P_1) \leq U(f, P_2)$

Definition : Upper & Lower Integrals

Let $f : [a, b] \to \mathbb{R}$ is a bounded function, let $\mathcal{P}$ be the set of all partitions of $[a, b]$. The upper integral of f is defined to be

$$U(f) = \inf\{U(f, P) : P \in \mathcal{P}\}$$

and the lower integral is:

$$L(f) = \sup\{L(f, P) : P \in \mathcal{P}\}$$

Lemma

Let $f : [a, b] \to \mathbb{R}$ be bounded with $m \leq f(x) \leq M$ for all $x \in [a, b]$, then

$$m(b - a) \leq L(f) \leq U(f) \leq M(b - a)$$

Definition : Darboux Integrable

A bounded function $f : [a, b] \to \mathbb{R}$ is (Darboux) Integrable if $L(f) = U(f)$. When this happens, we denote the common value by

$$\int_a^b f \qquad \text{or} \qquad \int_a^b f(x)\, dx$$

Proposition

Assume a bounded function $f : [a, b] \to \mathbb{R}$ is integrable. If $f \geq 0$ for all $x \in [a, b]$ then

$$\int_a^b f(x)\ dx \geq 0$$

Theorem

Assume $f : [a, b] \to \mathbb{R}$ is bounded. Then f is integrable if and only if for all $\varepsilon > 0$, there exists a partition P_ε of $[a, b]$ such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon$$

Corollary

If $f : [a, b] \to \mathbb{R}$ is integrable then there exists a sequence of partitions P_n of $[a, b]$ such that

$$\lim_{n\to\infty} \left[U(f, P_n) - L(f, P_n)\right] = 0$$

Theorem : (8.16)

If $f : [a, b] \to \mathbb{R}$ is continuous, then it is integrable.

Porism : (8.18)

Let $c \in [a, b]$ and suppose $f : [a, b] \to \mathbb{R}$ with

$$f(x) = \begin{cases} 1 & \text{if } x \neq c \\ 0 & \text{if } x = c \end{cases}$$

then f is integrable.

Definition : Length of Intervals

The length of an interval $[a, b]$ is $b - a$, denoted by $\mathcal{L}([a, b])$.

The length of the following intervals are also $b - a$: (a, b), $(a, b]$, $[a, b)$.

Intervals with infinity have length ∞.

Theorem

Assume $f : [a, b] \to \mathbb{R}$ is bounded, and let D be the set of points at which f is discontinuous. Then f is integrable if and only if D has measure 0.

Proposition

Assume $f : [a, b] \to \mathbb{R}$ is integrable. If $a < c < b$, then

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

Proposition

Assume $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are integrable, then

1. For all $k \in \mathbb{R}$, kf is also integrable on $[a, b]$ and $\int_a^b kf(x)\ dx = k \int_a^b f(x)\ dx$
2. The function $f + g$ is also integrable on $[a, b]$ and $\int_a^b (f + g)(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$

Corollary

Suppose $f(x) \leq g(x)$ for all $x \in [a, b]$ where f, g are both integrable. Then,

$$\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$$

Corollary

Suppose $f : [a, b] \to \mathbb{R}$ is integrable then $|f|$ is also integrable and

$$\left| \int_a^b f(x)\ dx \right| \leq \int_a^b |f(x)|\ dx$$

Definition

$$\int_a^a f(x)\ dx = 0$$

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$$

Proposition : (8.31)

If f is continuous on $[a, b]$, then there is some $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\ dx$$

Theorem : (Fundamental Theorem of Calculus - FTC)

1. If $f : [a, b] \to \mathbb{R}$ is integrable and $F : [a, b] \to \mathbb{R}$ satisfies $F'(x) = f(x)$ for all $x \in [a, b]$, then
$$\int_a^b f(x)\ dx = F(b) - F(a)$$

2. Let $g : [a, b] \to \mathbb{R}$ be integrable and define $G : [a, b] \to \mathbb{R}$ by $G(x) = \int_a^x g(t)\ dt$, then G is continuous. Moreover, if g is continuous then G is differentiable and $G'(x) = g(x)$.